

FRACTIONAL HERGLOTZ VARIATIONAL PROBLEMS OF VARIABLE ORDER

DINA TAVARES*

ESECS, Polytechnic Institute of Leiria, 2411–901 Leiria, Portugal
and

Center for Research and Development in Mathematics and Applications (CIDMA)
Department of Mathematics, University of Aveiro, 3810–193 Aveiro, Portugal

RICARDO ALMEIDA AND DELFIM F. M. TORRES

Center for Research and Development in Mathematics and Applications (CIDMA)
Department of Mathematics, University of Aveiro, 3810–193 Aveiro, Portugal

ABSTRACT. We study fractional variational problems of Herglotz type of variable order. Necessary optimality conditions, described by fractional differential equations depending on a combined Caputo fractional derivative of variable order, are proved. Two different cases are considered: the fundamental problem, with one independent variable, and the general case, with several independent variables. We end with some illustrative examples of the results of the paper.

1. Introduction. The theory of fractional calculus is an extension of ordinary calculus that considers integrals and derivatives of arbitrary real or complex order. Although its birth goes back to Euler, fractional calculus has gained a great importance only in recent decades, with the applicability of such operators for the efficient dynamic modeling of some real phenomena [2, 18]. More recently, a general theory of fractional calculus was presented, where the order of the fractional operators is not constant in time [12]. This is a natural extension, since fractional derivatives are nonlocal operators and contain memory. Therefore, it is reasonable that the order of the derivative may vary along time.

The variational problem of Herglotz is a generalization of the classical variational problem [16, 17]. It allows us to describe nonconservative processes, even in the case when the Lagrangian is autonomous (that is, when the Lagrangian does not depend explicitly on time). In contrast to the calculus of variations, where the cost functional is given by an integral depending only on time, space and velocity, in the Herglotz variational problem the model is given by a differential equation involving the derivative of the objective functional z and the Lagrange function depends on time, trajectories x and z and on the derivative of x . The problem of Herglotz was

2010 *Mathematics Subject Classification.* Primary: 26A33, 34A08; Secondary: 49K05, 49K10.

Key words and phrases. Fractional calculus, variable order, Herglotz variational problems.

This work is part of first author's Ph.D., which is carried out at the University of Aveiro under the Doctoral Programme *Mathematics and Applications* of Universities of Aveiro and Minho. It was supported by Portuguese funds through CIDMA and *The Portuguese Foundation for Science and Technology* (FCT), within project UID/MAT/04106/2013. Tavares was also supported by FCT through the Ph.D. fellowship SFRH/BD/42557/2007.

*Corresponding author: Dina Tavares (dtavares@ipleiria.pt).

posed by Herglotz himself in 1930 [7], but only in 1996, with the works [5, 6], it has gained a wide attention from the mathematical community. Indeed, since 1996, several papers were devoted to this subject: see [1, 3, 4, 13, 14, 15, 16, 17] and references therein.

2. Preliminaries. In this section we present some needed concepts and results.

2.1. The fractional calculus of variable order. We deal with fractional operators of variable fractional order on two variables, with range on the open interval $(0, 1)$, that is, the order is a function $\alpha : [a, b]^2 \rightarrow (0, 1)$. Given a function $x : [a, b] \rightarrow \mathbb{R}$, we present two different concepts of fractional derivatives of x . First, we recall the definition of fractional integral [10].

Definition 2.1. The left Riemann–Liouville fractional integral of order $\alpha(\cdot, \cdot)$ of x is defined by

$${}_a I_t^{\alpha(\cdot, \cdot)} x(t) = \int_a^t \frac{1}{\Gamma(\alpha(t, \tau))} (t - \tau)^{\alpha(t, \tau) - 1} x(\tau) d\tau$$

and the right Riemann–Liouville fractional integral of x by

$${}_t I_b^{\alpha(\cdot, \cdot)} x(t) = \int_t^b \frac{1}{\Gamma(\alpha(\tau, t))} (\tau - t)^{\alpha(\tau, t) - 1} x(\tau) d\tau.$$

For fractional derivatives, we consider two types: Riemann–Liouville and Caputo fractional derivatives.

Definition 2.2. The left Riemann–Liouville fractional derivative of order $\alpha(\cdot, \cdot)$ of x is defined by

$${}_a D_t^{\alpha(\cdot, \cdot)} x(t) = \frac{d}{dt} \int_a^t \frac{1}{\Gamma(1 - \alpha(t, \tau))} (t - \tau)^{-\alpha(t, \tau)} x(\tau) d\tau$$

and the right Riemann–Liouville fractional derivative of x by

$${}_t D_b^{\alpha(\cdot, \cdot)} x(t) = \frac{d}{dt} \int_t^b \frac{-1}{\Gamma(1 - \alpha(\tau, t))} (\tau - t)^{-\alpha(\tau, t)} x(\tau) d\tau.$$

Definition 2.3. The left Caputo fractional derivative of order $\alpha(\cdot, \cdot)$ of x is defined by

$${}_a^C D_t^{\alpha(\cdot, \cdot)} x(t) = \int_a^t \frac{1}{\Gamma(1 - \alpha(t, \tau))} (t - \tau)^{-\alpha(t, \tau)} x^{(1)}(\tau) d\tau$$

and the right Caputo fractional derivative of x by

$${}_t^C D_b^{\alpha(\cdot, \cdot)} x(t) = \int_t^b \frac{-1}{\Gamma(1 - \alpha(\tau, t))} (\tau - t)^{-\alpha(\tau, t)} x^{(1)}(\tau) d\tau.$$

Motivated by the works [8, 9], we consider here a generalization of previous concepts by introducing a linear combination of the fractional derivatives of variable fractional order.

Definition 2.4. Let $\alpha, \beta : [a, b]^2 \rightarrow (0, 1)$ be two functions and $\gamma = (\gamma_1, \gamma_2) \in [0, 1]^2$ a vector. The combined Riemann–Liouville fractional derivative of function x is defined by

$$D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) = \gamma_1 {}_a D_t^{\alpha(\cdot, \cdot)} x(t) + \gamma_2 {}_t D_b^{\beta(\cdot, \cdot)} x(t). \quad (1)$$

Similarly, the combined Caputo fractional derivative of function x is defined by

$${}_a^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) = \gamma_1 {}_a^C D_t^{\alpha(\cdot, \cdot)} x(t) + \gamma_2 {}_t^C D_b^{\beta(\cdot, \cdot)} x(t). \quad (2)$$

When dealing with variational problems and necessary optimality conditions, an important ingredient is always an integration by parts formula. Here, we present two such formulas, involving the Caputo fractional derivative of variable order.

Theorem 2.5 (See [11, Theorem 3.2]). *If $x, y \in C^1[a, b]$, then*

$$\int_a^b y(t) {}^C D_t^{\alpha(\cdot, \cdot)} x(t) dt = \int_a^b x(t) {}_t D_b^{\alpha(\cdot, \cdot)} y(t) dt + \left[x(t) {}_t I_b^{1-\alpha(\cdot, \cdot)} y(t) \right]_{t=a}^{t=b}$$

and

$$\int_a^b y(t) {}_t D_b^{\alpha(\cdot, \cdot)} x(t) dt = \int_a^b x(t) {}_a D_t^{\alpha(\cdot, \cdot)} y(t) dt - \left[x(t) {}_a I_t^{1-\alpha(\cdot, \cdot)} y(t) \right]_{t=a}^{t=b}.$$

To end this short introduction to the fractional calculus of variable order, we introduce one more notation. The dual fractional derivative of (1) is defined by

$$D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} = \gamma_2 {}_a D_t^{\beta(\cdot, \cdot)} + \gamma_1 {}_t D_T^{\alpha(\cdot, \cdot)},$$

where $\bar{\gamma} = (\gamma_2, \gamma_1)$ and $T \in [a, b]$ is the final time of the problem under consideration (see (3) below). The dual fractional derivative of (2) is defined similarly.

2.2. The fractional calculus of variations. Let D denote the linear subspace of $C^1([a, b]) \times [a, b]$ defined by

$$D = \left\{ (x, t) \in C^1([a, b]) \times [a, b] : {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) \text{ exists and is continuous on } [a, b] \right\}.$$

We endow D with the following norm:

$$\|(x, t)\| = \max_{a \leq t \leq b} |x(t)| + \max_{a \leq t \leq b} \left| {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) \right| + |t|.$$

To fix notation, throughout the text we denote by $\partial_i \psi$ the partial derivative of a function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ with respect to its i th argument, $i = 1, \dots, n$. For simplicity of notation, we also introduce the operator $[\cdot]_{\gamma}^{\alpha, \beta}$ defined by

$$[x]_{\gamma}^{\alpha, \beta}(t) = \left(t, x(t), {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) \right).$$

Let L be a Lagrangian $L : C^1([a, b] \times \mathbb{R}^2) \rightarrow \mathbb{R}$. Consider the following problem of the calculus of variations: minimize functional $\mathcal{J} : D \rightarrow \mathbb{R}$ with

$$\mathcal{J}(x, T) = \int_a^T L[x]_{\gamma}^{\alpha, \beta}(t) dt + \phi(T, x(T)) \quad (3)$$

over all $(x, T) \in D$ satisfying the initial condition $x(a) = x_a$, for a given $x_a \in \mathbb{R}$. The terminal time T and the terminal state $x(T)$ are considered here free. The terminal cost function $\phi : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is at least of class C^1 .

Theorem 2.6 (See [19]). *If (x, T) is a minimizer of functional (3) on D , then (x, T) satisfies the fractional differential equations*

$$\partial_2 L[x]_{\gamma}^{\alpha, \beta}(t) + D_{\bar{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) = 0 \quad (4)$$

on $[a, T]$ and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) - {}_T D_t^{\beta(\cdot, \cdot)} \partial_3 L[x]_{\gamma}^{\alpha, \beta}(t) \right) = 0 \quad (5)$$

on $[T, b]$. Moreover, the following transversality conditions hold:

$$\begin{cases} L[x]_\gamma^{\alpha, \beta}(T) + \partial_1 \phi(T, x(T)) + \partial_2 \phi(T, x(T))x'(T) = 0, \\ \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} + \partial_2 \phi(T, x(T)) = 0, \\ \gamma_2 \left[{}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=b} = 0. \end{cases} \quad (6)$$

We can rewrite the transversality conditions (6), obtaining the next result.

Theorem 2.7 (See [19]). *If (x, T) is a minimizer of functional (3) on D , then the fractional Euler–Lagrange equations (4) and (5) are satisfied together with the following transversality conditions:*

$$\begin{cases} L[x]_\gamma^{\alpha, \beta}(T) + \partial_1 \phi(T, x(T)) \\ \quad + x'(T) \left[\gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) - \gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} = 0, \\ \left[\gamma_1 {}_t I_T^{1-\alpha(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=T} + \partial_2 \phi(T, x(T)) = 0, \\ \gamma_2 \left[{}_T I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) - {}_a I_t^{1-\beta(\cdot, \cdot)} \partial_3 L[x]_\gamma^{\alpha, \beta}(t) \right]_{t=b} = 0. \end{cases}$$

3. Herglotz’s variational principle. In this section we present a fractional variational principle of Herglotz depending on Caputo fractional derivatives.

Let $\alpha, \beta : [a, b]^2 \rightarrow (0, 1)$ be two functions. The fractional Herglotz variational problem that we study consists in the determination of trajectories $x \in C^1([a, b])$, satisfying a given initial condition $x(a) = x_a \in \mathbb{R}$, and a real $T \in [a, b]$ that extremize the value of $z(T)$, where z satisfies the following differential equation with dependence on a combined Caputo fractional derivative operator:

$$\dot{z}(t) = L\left(t, x(t), {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t), z(t)\right), \quad t \in [a, b], \quad (7)$$

subject to the initial condition

$$z(a) = z_a, \quad (8)$$

where z_a is a given real number. In the sequel, we use the auxiliary notation

$$[x, z]_\gamma^{\alpha, \beta}(t) = \left(t, x(t), {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t), z(t)\right).$$

The Lagrangian L is assumed to satisfy the following hypothesis:

1. $L \in C^1([a, b] \times \mathbb{R}^3, \mathbb{R})$,
2. $t \rightarrow \lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)$ is such that ${}_T D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t))$, ${}_a D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t))$ and $D_{\overline{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t))$ exist and are continuous on $[a, b]$, where

$$\lambda(t) = \exp\left(-\int_a^t \partial_4 L[x, z]_\gamma^{\alpha, \beta}(\tau) d\tau\right).$$

The following result gives necessary conditions of Euler–Lagrange type for an admissible function x to be solution of the problem.

Theorem 3.1. *Let $x \in C^1([a, b])$ be such that z defined by (7) subject to the initial condition (8) has an extremum. Then, (x, z) satisfies the fractional differential equations*

$$\partial_2 L[x, z]_\gamma^{\alpha, \beta}(t) \lambda(t) + D_{\overline{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) = 0 \quad (9)$$

on $[a, T]$ and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) - {}_T D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) \right) = 0 \quad (10)$$

on $[T, b]$. Moreover, the following transversality conditions are satisfied:

$$\begin{cases} \left[\gamma_1 {}_T I_t^{1-\alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) - \gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) \right]_{t=T} = 0, \\ \left[\gamma_2 {}_T I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) - {}_a I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) \right]_{t=b} = 0. \end{cases} \quad (11)$$

If $T < b$, then $L[x, z]_\gamma^{\alpha, \beta}(T) = 0$.

Proof. Let x be a solution to the problem and consider an admissible variation of x , $\bar{x} = x + \epsilon h$, where $h \in C^1([a, b])$ is an arbitrary perturbation curve and $\epsilon \in \mathbb{R}$ represents a small number ($\epsilon \rightarrow 0$). The constraint $x(a) = x_a$ implies that all admissible variations must fulfill the condition $h(a) = 0$. On the other hand, consider an admissible variation of z , $\bar{z} = z + \epsilon \theta$, where θ is a perturbation curve (not arbitrary) such that

1. $\theta(a) = 0$, so that $z(a) = z_a$,
2. $\theta(T) = 0$, because $z(T)$ is a maximum ($\bar{z}(T) - z(T) \leq 0$) or a minimum ($\bar{z}(T) - z(T) \geq 0$),
3. $\theta(t) = \frac{d}{d\epsilon} z(\bar{x}, t) \Big|_{\epsilon=0}$, so that the variation satisfies equation (7).

Differentiating θ with respect to t , we obtain that

$$\begin{aligned} \frac{d}{dt} \theta(t) &= \frac{d}{dt} \frac{d}{d\epsilon} z(\bar{x}, t) \Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \frac{d}{dt} z(\bar{x}, t) \Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} L \left(t, x(t) + \epsilon h(t), {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) + \epsilon {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t), z(t) \right) \Big|_{\epsilon=0} \end{aligned}$$

and rewriting this relation, we obtain the following differential equation for θ :

$$\dot{\theta}(t) - \partial_4 L[x, z]_\gamma^{\alpha, \beta}(t) \theta(t) = \partial_2 L[x, z]_\gamma^{\alpha, \beta}(t) h(t) + \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t) {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t).$$

Considering $\lambda(t) = \exp \left(- \int_a^t \partial_4 L[x, z]_\gamma^{\alpha, \beta}(\tau) d\tau \right)$, we obtain the solution for the last differential equation:

$$\theta(T) \lambda(T) - \theta(a) = \int_a^T \left(\partial_2 L[x, z]_\gamma^{\alpha, \beta}(t) h(t) + \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t) {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t) \right) \lambda(t) dt.$$

By hypothesis, $\theta(a) = 0$. If x is such that $z(x, t)$ defined by (7) attains an extremum at $t = T$, then $\theta(T)$ is identically zero. Hence, we get

$$\int_a^T \left(\partial_2 L[x, z]_\gamma^{\alpha, \beta}(t) h(t) + \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t) {}^C D_\gamma^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} h(t) \right) \lambda(t) dt = 0. \quad (12)$$

Considering only the second term in (12), and the definition of combined Caputo derivative, we obtain that

$$\begin{aligned}
& \int_a^T \lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t) \left(\gamma_1 {}^C D_t^{\alpha(\cdot, \cdot)} h(t) + \gamma_2 {}^C D_b^{\beta(\cdot, \cdot)} h(t) \right) dt \\
&= \gamma_1 \int_a^T \lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t) {}^C D_t^{\alpha(\cdot, \cdot)} h(t) dt \\
&+ \gamma_2 \left[\int_a^b \lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t) {}^C D_b^{\beta(\cdot, \cdot)} h(t) dt - \int_T^b \lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t) {}^C D_b^{\beta(\cdot, \cdot)} h(t) dt \right] \\
&= \star.
\end{aligned}$$

Using Theorem 2.5, and considering $\overline{\gamma} = (\gamma_2, \gamma_1)$, we get

$$\begin{aligned}
\star &= \int_a^T h(t) D_{\overline{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) dt \\
&+ \int_T^b \gamma_2 h(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) - {}_T D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) \right] dt \\
&+ h(T) \left[\gamma_1 {}_T I_T^{1-\alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) - \gamma_2 {}_T I_T^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) \right]_{t=T} \\
&+ h(b) \gamma_2 \left[{}_T I_T^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) - {}_a I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) \right]_{t=b}.
\end{aligned}$$

Substituting this relation into expression (12), we obtain

$$\begin{aligned}
& \int_a^T h(t) \left[\partial_2 L[x, z]_\gamma^{\alpha, \beta}(t) \lambda(t) + D_{\overline{\gamma}}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) \right] dt \\
&+ \int_T^b \gamma_2 h(t) \left[{}_a D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) - {}_T D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) \right] dt \\
&+ h(T) \left[\gamma_1 {}_T I_T^{1-\alpha(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) - \gamma_2 {}_T I_T^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) \right]_{t=T} \\
&+ h(b) \gamma_2 \left[{}_T I_T^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) - {}_a I_t^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)) \right]_{t=b} = 0.
\end{aligned}$$

With appropriate choices for the variations $h(\cdot)$, we get the Euler–Lagrange equations (9)–(10) and the transversality conditions (11). \square

Remark 1. If $\alpha(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ tend to 1, and if the Lagrangian L is of class C^2 , then the first Euler–Lagrange equation (9) becomes

$$\partial_2 L[x, z]_\gamma^{\alpha, \beta}(t) \lambda(t) + (\gamma_2 - \gamma_1) \frac{d}{dt} [\lambda(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t)] = 0.$$

Differentiating and considering the derivative of the lambda function, we get

$$\begin{aligned}
& \lambda(t) \left[\partial_2 L[x, z]_\gamma^{\alpha, \beta}(t) \right. \\
& \quad \left. + (\gamma_2 - \gamma_1) \left[-\partial_4 L[x, z]_\gamma^{\alpha, \beta}(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t) + \frac{d}{dt} \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t) \right] \right] = 0.
\end{aligned}$$

As $\lambda(t) > 0$ for all t , we deduce that

$$\partial_2 L[x, z]_\gamma^{\alpha, \beta}(t) + (\gamma_2 - \gamma_1) \left[\frac{d}{dt} \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t) - \partial_4 L[x, z]_\gamma^{\alpha, \beta}(t) \partial_3 L[x, z]_\gamma^{\alpha, \beta}(t) \right] = 0.$$

4. The case of several independent variables. We now obtain a generalization of Herglotz's principle of Section 3 for problems involving $n + 1$ independent variables. Define $\Omega = \prod_{i=1}^n [a_i, b_i]$, with $n \in \mathbb{N}$, $P = [a, b] \times \Omega$ and consider the vector $s = (s_1, s_2, \dots, s_n) \in \Omega$. The new problem consists in determining the trajectories $x \in C^1(P)$ that give an extremum to $z[x, T]$, where the functional z satisfies the differential equation

$$\dot{z}(t) = \int_{\Omega} L \left(t, s, x(t, s), {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t, s), \right. \\ \left. {}^C D_{\gamma^1}^{\alpha_1(\cdot, \cdot), \beta_1(\cdot, \cdot)} x(t, s), \dots, {}^C D_{\gamma^n}^{\alpha_n(\cdot, \cdot), \beta_n(\cdot, \cdot)} x(t, s), z(t) \right) d^n s \quad (13)$$

subject to the constraint

$$x(t, s) = g(t, s) \quad \text{for all } (t, s) \in \partial P, \quad (14)$$

where ∂P is the boundary of P and g is a given function $g : \partial P \rightarrow \mathbb{R}$. Assume that

1. $\alpha, \alpha_i, \beta, \beta_i : [a, b]^2 \rightarrow (0, 1)$ with $i = 1, \dots, n$,
2. $\gamma, \gamma^1, \dots, \gamma^n \in [0, 1]^2$,
3. $d^n s = ds_1 \dots ds_n$,
4. ${}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t, s), {}^C D_{\gamma^1}^{\alpha_1(\cdot, \cdot), \beta_1(\cdot, \cdot)} x(t, s), \dots, {}^C D_{\gamma^n}^{\alpha_n(\cdot, \cdot), \beta_n(\cdot, \cdot)} x(t, s)$ exist and are continuous functions,
5. the Lagrangian $L : P \times \mathbb{R}^{n+3} \rightarrow \mathbb{R}$ is of class C^1 .

Remark 2. By ${}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t, s)$ we mean the Caputo fractional derivative with respect to the independent variable t and by ${}^C D_{\gamma^i}^{\alpha_i(\cdot, \cdot), \beta_i(\cdot, \cdot)} x(t, s)$ we mean the Caputo fractional derivative with respect to the independent variable s_i , $i = 1, \dots, n$.

In the sequel, we use the auxiliary notation

$$[x, z]_{n, \gamma}^{\alpha, \beta}(t, s) = \left(t, s, x(t, s), {}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t, s), {}^C D_{\gamma^1}^{\alpha_1(\cdot, \cdot), \beta_1(\cdot, \cdot)} x(t, s), \right. \\ \left. \dots, {}^C D_{\gamma^n}^{\alpha_n(\cdot, \cdot), \beta_n(\cdot, \cdot)} x(t, s), z(t) \right).$$

Consider the function

$$\lambda(t) = \exp \left(- \int_a^t \int_{\Omega} \partial_{2n+4} [x, z]_{n, \gamma}^{\alpha, \beta}(\tau, s) d^n s d\tau \right).$$

Theorem 4.1. *If (x, z) is an extremizer to the functional (13), then (x, z) satisfies the fractional differential equation*

$$\partial_{n+2} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s) \lambda(t) + D_{\gamma}^{\beta(\cdot, \cdot), \alpha(\cdot, \cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) \\ + \sum_{i=1}^n D_{\gamma^i}^{\beta_i(\cdot, \cdot), \alpha_i(\cdot, \cdot)} (\lambda(t) \partial_{n+3+i} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) = 0 \quad (15)$$

on $[a, T] \times \Omega$ and

$$\gamma_2 \left({}_a D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) - {}_T D_t^{\beta(\cdot, \cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) \right) = 0 \quad (16)$$

on $[T, b] \times \Omega$. Moreover, (x, z) satisfies the transversality condition

$$\left[\gamma_1 {}_T I_T^{1-\alpha(\cdot, \cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) \right. \\ \left. - \gamma_2 {}_T I_T^{1-\beta(\cdot, \cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n, \gamma}^{\alpha, \beta}(t, s)) \right]_{t=T} = 0, \quad s \in \Omega. \quad (17)$$

If $T < b$, then $\int_{\Omega} L[x, z]_{n,\gamma}^{\alpha,\beta}(T, s) d^n s = 0$.

Proof. Let x be a solution to the problem. Consider an admissible variation of x , $\bar{x}(t, s) = x(t, s) + \epsilon h(t, s)$, where $h \in C^1(P)$ is an arbitrary perturbing curve and $\epsilon \in \mathbb{R}$ is such that $|\epsilon| \ll 1$. Consequently, from the boundary condition (14), $h(t, s) = 0$ for all $(t, s) \in \partial P$. On the other hand, consider an admissible variation of z , $\bar{z} = z + \epsilon \theta$, where θ is a perturbing curve such that $\theta(a) = 0$ and

$$\theta(t) = \left. \frac{d}{d\epsilon} z(\bar{x}, t) \right|_{\epsilon=0}.$$

Differentiating $\theta(t)$ with respect to t , we obtain that

$$\begin{aligned} \frac{d}{dt} \theta(t) &= \left. \frac{d}{dt} \frac{d}{d\epsilon} z(\bar{x}, t) \right|_{\epsilon=0} \\ &= \left. \frac{d}{d\epsilon} \frac{d}{dt} z(\bar{x}, t) \right|_{\epsilon=0} \\ &= \left. \frac{d}{d\epsilon} \int_{\Omega} L[\bar{x}, z]_{n,\gamma}^{\alpha,\beta}(t, s) d^n s \right|_{\epsilon=0}. \end{aligned}$$

We conclude that

$$\begin{aligned} \dot{\theta}(t) &= \int_{\Omega} \left(\partial_{n+2} L[x, z]_{n,\gamma}^{\alpha,\beta}(t) h(t, s) + \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)^C D_{\gamma}^{\alpha(\cdot,\cdot), \beta(\cdot,\cdot)} h(t, s) \right. \\ &\quad \left. + \sum_{i=1}^n \partial_{n+3+i} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)^C D_{\gamma^i}^{\alpha_i(\cdot,\cdot), \beta_i(\cdot,\cdot)} h(t, s) + \partial_{2n+4} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) \theta(t) \right) d^n s. \end{aligned}$$

To simplify the notation, define

$$B(t) = \int_{\Omega} \partial_{2n+4} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) d^n s$$

and

$$\begin{aligned} A(t) &= \int_{\Omega} \left(\partial_{n+2} L[x, z]_{n,\gamma}^{\alpha,\beta}(t) h(t, s) + \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)^C D_{\gamma}^{\alpha(\cdot,\cdot), \beta(\cdot,\cdot)} h(t, s) \right. \\ &\quad \left. + \sum_{i=1}^n \partial_{n+3+i} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)^C D_{\gamma^i}^{\alpha_i(\cdot,\cdot), \beta_i(\cdot,\cdot)} h(t, s) \right) d^n s. \end{aligned}$$

Then, we obtain the linear differential equation

$$\dot{\theta}(t) - B(t)\theta(t) = A(t),$$

whose solution is

$$\theta(T)\lambda(T) - \theta(a) = \int_a^T A(t)\lambda(t) dt.$$

Since $\theta(a) = \theta(T) = 0$, we get

$$\int_a^T A(t)\lambda(t) dt = 0. \quad (18)$$

Considering only the second term in (18), we can write

$$\begin{aligned} & \int_a^T \int_{\Omega} \lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) \left(\gamma_1^C D_t^{\alpha(\cdot,\cdot)} h(t, s) + \gamma_2^C D_b^{\beta(\cdot,\cdot)} h(t, s) \right) d^n s dt \\ &= \gamma_1 \int_a^T \int_{\Omega} \lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) {}_a^C D_t^{\alpha(\cdot,\cdot)} h(t, s) d^n s dt \\ &+ \gamma_2 \left[\int_a^b \int_{\Omega} \lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) {}_t^C D_b^{\beta(\cdot,\cdot)} h(t, s) d^n s dt \right. \\ &\quad \left. - \int_T^b \int_{\Omega} \lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) {}_t^C D_b^{\beta(\cdot,\cdot)} h(t, s) d^n s dt \right]. \end{aligned}$$

Let $\overline{\gamma} = (\gamma_2, \gamma_1)$. Integrating by parts (cf. Theorem 2.5) and since $h(a, s) = 0$ and $h(b, s) = 0$ for all $s \in \Omega$, we obtain the following expression:

$$\begin{aligned} & \int_a^T \int_{\Omega} h(t, s) D_{\overline{\gamma}}^{\beta(\cdot,\cdot), \alpha(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) d^n s dt \\ &+ \gamma_2 \int_T^b \int_{\Omega} h(t, s) \left[{}_a D_t^{\beta(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right. \\ &\quad \left. - {}_T D_t^{\beta(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right] d^n s dt \\ &+ \int_{\Omega} h(T, s) \left[\gamma_1 {}_T I_T^{1-\alpha(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right. \\ &\quad \left. - \gamma_2 {}_T I_t^{1-\beta(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right]_{t=T} d^n s. \end{aligned}$$

Doing similarly for the $(i+2)$ th term of (18), $i = 1, \dots, n$, letting $\overline{\gamma}^i = (\gamma_2^i, \gamma_1^i)$, and since $h(t, a_i) = h(t, b_i) = 0$ for all $t \in [a, b]$, we obtain

$$\begin{aligned} & \int_a^T \int_{\Omega} \lambda(t) \partial_{n+3+i} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) \left(\gamma_1^i {}_a^C D_{s_i}^{\alpha_i(\cdot,\cdot)} h(t, s) + \gamma_2^i {}_b^C D_{b_i}^{\beta_i(\cdot,\cdot)} h(t, s) \right) d^n s dt \\ &= \int_a^T \int_{\Omega} h(t, s) D_{\overline{\gamma}^i}^{\beta_i(\cdot,\cdot), \alpha_i(\cdot,\cdot)} (\lambda(t) \partial_{n+3+i} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) d^n s dt. \end{aligned}$$

Substituting these relations into (18), we deduce that

$$\begin{aligned} & \int_a^T \int_{\Omega} h(t, s) \left[\partial_{n+2} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s) \lambda(t) + D_{\overline{\gamma}}^{\beta(\cdot,\cdot), \alpha(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right. \\ &+ \sum_{i=1}^n D_{\overline{\gamma}^i}^{\beta_i(\cdot,\cdot), \alpha_i(\cdot,\cdot)} (\lambda(t) \partial_{n+3+i} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) d^n s dt \\ &+ \gamma_2 \int_T^b \int_{\Omega} h(t, s) \left[{}_a D_t^{\beta(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right. \\ &\quad \left. - {}_T D_t^{\beta(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right] d^n s dt \\ &+ \int_{\Omega} h(T, s) \left[\gamma_1 {}_T I_T^{1-\alpha(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right. \\ &\quad \left. - \gamma_2 {}_T I_t^{1-\beta(\cdot,\cdot)} (\lambda(t) \partial_{n+3} L[x, z]_{n,\gamma}^{\alpha,\beta}(t, s)) \right]_{t=T} d^n s. \end{aligned}$$

We get the Euler–Lagrange equations (15)–(16) and the transversality condition (17) with appropriate choices of h . \square

5. Illustrative examples. We present three examples.

Example 1. Consider

$$\begin{aligned}\dot{z}(t) &= \left({}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t)\right)^2 + z(t) + t^2 - 1, \quad t \in [0, 3], \\ x(0) &= 1, \quad z(0) = 0.\end{aligned}\tag{19}$$

In this case, $\lambda(t) = \exp(-t)$. The necessary optimality conditions (9)–(10) of Theorem 3.1 hold for $\bar{x}(t) \equiv 1$. If we replace x by \bar{x} in (19), we obtain

$$\begin{aligned}\dot{z}(t) - z(t) &= t^2 - 1, \quad t \in [0, 3], \\ z(0) &= 0,\end{aligned}$$

whose solution is

$$z(t) = \exp(t) - (t+1)^2.\tag{20}$$

The last transversality condition of Theorem 3.1 asserts that

$$L[\bar{x}, z]_{\gamma}^{\alpha, \beta}(T) = 0 \Leftrightarrow \exp(T) - 2T - 2 = 0,$$

whose solution is approximately

$$T \approx 1.67835.$$

We remark that z (20) actually attains a minimum value at this point (see Figure 1, (a)):

$$z(1.67835) \approx -1.81685.$$

Example 2. Consider now

$$\begin{aligned}\dot{z}(t) &= (t-1)(x^2(t) + z^2(t) + 1), \quad t \in [0, 3], \\ x(0) &= 0, \quad z(0) = 0.\end{aligned}\tag{21}$$

Since the first Euler–Lagrange equation (9) reads

$$(t-1)x(t) = 0 \quad \forall t \in [0, T],$$

we see that $\bar{x}(t) \equiv 0$ is a solution of this equation. The second transversality condition of (11) asserts that, at $t = T$, we must have

$$L[\bar{x}, z]_{\gamma}^{\alpha, \beta}(t) = 0,$$

that is,

$$(t-1)(z^2(t) + 1) = 0,$$

and so $T = 1$ is a solution of this equation. Substituting x by \bar{x} in (21), we get

$$\begin{aligned}\dot{z}(t) &= (t-1)(z^2(t) + 1), \quad t \in [0, 3], \\ z(0) &= 0.\end{aligned}$$

The solution to this Cauchy problem is the function

$$z(t) = \tan\left(\frac{t^2}{2} - t\right)$$

(see Figure 1, (b)) and the minimum value is

$$z(1) = \tan\left(-\frac{1}{2}\right).$$

Example 3. For our last example, consider

$$\begin{aligned} \dot{z}(t) &= \left({}^C D_{\gamma}^{\alpha(\cdot, \cdot), \beta(\cdot, \cdot)} x(t) - f(t) \right)^2 + t^2 - 1, \quad t \in [0, 3], \\ x(0) &= 0, \quad z(0) = 0, \end{aligned} \quad (22)$$

where

$$f(t) := \frac{t^{1-\alpha(t)}}{2\Gamma(2-\alpha(t))} - \frac{(3-t)^{1-\beta(t)}}{2\Gamma(2-\beta(t))}.$$

In this case, $\lambda(t) \equiv 1$. We intend to find a pair (x, z) , satisfying all the conditions in (22), for which $z(T)$ attains a minimum value. It is easy to verify that $\bar{x}(t) = t$ and $T = 1$ satisfy the necessary conditions given by Theorem 3.1. Replacing x by \bar{x} in system (22), we get a Cauchy problem of form

$$\begin{aligned} \dot{z}(t) &= t^2 - 1, \quad t \in [0, 3], \\ z(0) &= 0, \end{aligned}$$

whose solution is

$$z(t) = \frac{t^3}{3} - t.$$

Observe that this function attains a minimum value at $T = 1$, which is $z(1) = -2/3$ (Figure 1, (c)).

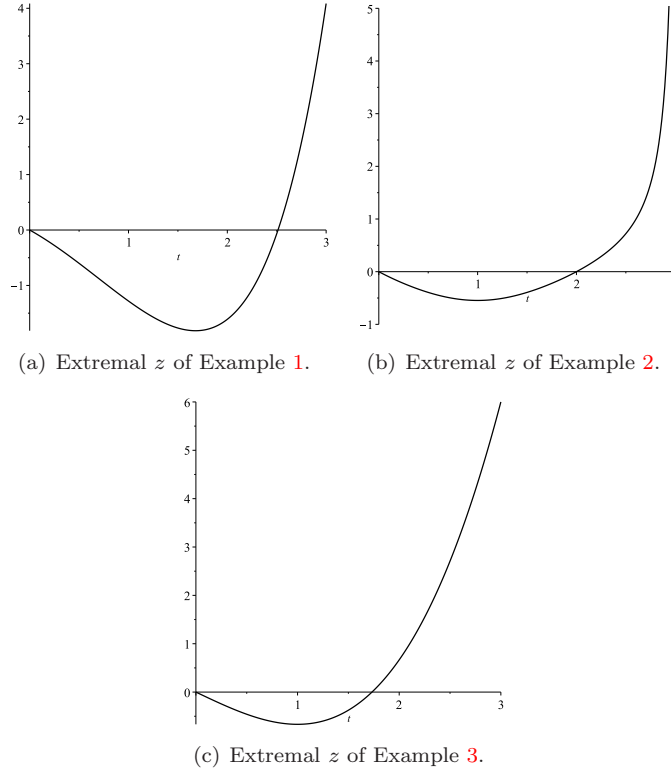


FIGURE 1. Graphics of function $z(\bar{x}, t)$.

Acknowledgements. The authors are grateful to two anonymous referees for their comments.

REFERENCES

- [1] R. Almeida and A. B. Malinowska, Fractional variational principle of Herglotz, *Discrete Contin. Dyn. Syst. Ser. B* **19** (2014), no. 8, 2367–2381. [arXiv:1406.0717](#)
- [2] F. M. Coimbra, C. M. Soon and M. H. Kobayashi, The variable viscoelasticity operator, *Annalender Physik* **14** (2005), 378–389.
- [3] B. Georgieva and R. Guenther, First Noether-type theorem for the generalized variational principle of Herglotz, *Topol. Methods Nonlinear Anal.* **20** (2002), no. 2, 261–273.
- [4] B. Georgieva, R. Guenther and T. Bodurov, Generalized variational principle of Herglotz for several independent variables, *J. Math. Phys.* **44** (2003), no. 9, 3911–3927.
- [5] R. B. Guenther, J. A. Gottsch and D. B. Kramer, The Herglotz algorithm for constructing canonical transformations, *SIAM Rev.* **38** (1996), no. 2, 287–293.
- [6] R. B. Guenther, C. M. Guenther and J. A. Gottsch, The Herglotz Lectures on Contact Transformations and Hamiltonian Systems, *Lecture Notes in Nonlinear Analysis*, Vol. 1, Juliusz Schauder Center for Nonlinear Studies, Nicholas Copernicus University, Toruń, 1996.
- [7] G. Herglotz, Berührungstransformationen, Lectures at the University of Göttingen, Göttingen, 1930.
- [8] A. B. Malinowska and D. F. M. Torres, Fractional calculus of variations for a combined Caputo derivative, *Fract. Calc. Appl. Anal.* **14** (2011), 523–537. [arXiv:1109.4664](#)
- [9] T. Odziejewicz, A. B. Malinowska and D. F. M. Torres, Fractional variational calculus with classical and combined Caputo derivatives, *Nonlinear Anal.* **75** (2012), 1507–1515. [arXiv:1101.2932](#)
- [10] T. Odziejewicz, A. B. Malinowska and D. F. M. Torres, Fractional variational calculus of variable order. In: *Advances in harmonic analysis and operator theory*, 291–301, Oper. Theory Adv. Appl., 229, Birkhäuser/Springer Basel AG, Basel, 2013. [arXiv:1110.4141](#)
- [11] T. Odziejewicz, A. B. Malinowska and D. F. M. Torres, Noether’s theorem for fractional variational problems of variable order, *Cent. Eur. J. Phys.* **11** (2013), 691–701. [arXiv:1303.4075](#)
- [12] S. G. Samko and B. Ross, Integration and differentiation to a variable fractional order, *Integral Transform. Spec. Funct.* **1** (1993), no. 4, 277–300.
- [13] S. P. S. Santos, N. Martins and D. F. M. Torres, Higher-order variational problems of Herglotz type, *Vietnam J. Math.* **42** (2014), no. 4, 409–419. [arXiv:1309.6518](#)
- [14] S. P. S. Santos, N. Martins and D. F. M. Torres, Variational problems of Herglotz type with time delay: DuBois-Reymond condition and Noether’s first theorem, *Discrete Contin. Dyn. Syst.* **35** (2015), no. 9, 4593–4610. [arXiv:1501.04873](#)
- [15] S. P. S. Santos, N. Martins and D. F. M. Torres, An optimal control approach to Herglotz variational problems. In: *Optimization in the Natural Sciences* (eds. A. Plakhov, T. Tchemisova and A. Freitas), Communications in Computer and Information Science, Vol. 499, Springer, 2015, 107–117. [arXiv:1412.0433](#)
- [16] S. P. S. Santos, N. Martins and D. F. M. Torres, Noether’s theorem for higher-order variational problems of Herglotz type, *Discrete Contin. Dyn. Syst.* **2015** (2015), Dynamical systems, differential equations and applications. 10th AIMS Conference. Suppl., 990–999. [arXiv:1507.05911](#)
- [17] S. P. S. Santos, N. Martins and D. F. M. Torres, Higher-order variational problems of Herglotz type with time delay, *Pure and Applied Functional Analysis* **1** (2016), no. 2, 291–307. [arXiv:1603.04034](#)
- [18] H. Sun, S. Hu, Y. Chen, W. Chen and Z. Yu, A dynamic-order fractional dynamic system, *Chinese Phys. Lett.* **30** (2013), 046601, 4 pp.
- [19] D. Tavares, R. Almeida and D. F. M. Torres, Optimality conditions for fractional variational problems with dependence on a combined caputo derivative of variable order, *Optim.* **64** (2015), 1381–1391. [arXiv:1501.02082](#)

Received July 29, 2016; Revised Feb 03, 2017; Accepted March 27, 2017.

E-mail address: dtavares@ipleiria.pt

E-mail address: ricardo.almeida@ua.pt

E-mail address: delfim@ua.pt